Computing the conical function $P^{\mu}_{-1/2+i\tau}(x)$

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Abstract

A stable computational scheme for the conical function $P^{\mu}_{-1/2+i\tau}(x)$ for $x > -1$, real τ, and $\mu \leq 0$ or $\mu \in \mathbb{N}$, is presented. The scheme combines uniform asymptotic expansions for large $|\mu|$ with the application of the three-term recurrence relation on the μ index in the direction of decreasing $|\mu|$ when $x > 0$. When $x < 0$ the conditioning of recursion is the opposite and conical functions can be computed in the direction of increasing $|\mu|$.

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1 Introduction

Conical functions appear in a large number of applications in engineering, applied physics [12] [11], quantum physics (related to the amplitude for

Yukawa potential scattering [3]) or cosmology [13], among others. Also, they are the kernel of the Mehler-Fock transform, which has numerous applications.

It appears that the only existing algorithm for conical functions is given by Kölbig [8], who computed the functions $P^0_{-1/2+i\tau}(x)$ and $P^1_{-1/2+i\tau}(x)$, $x>-1$ (see also [11]). In applications, however, in general one encounters values $P^{\mu}_{-1/2+i\tau}(x)$, where $\mu=m$ are integer numbers. We are providing a computational method for these parameter values, which uses uniform asymptotics for large $|m|$ combined with recurrence relations and numerical quadrature.

2 Basic definitions

The associated Legendre function $P_\nu^{-\mu}(x)$ for real $x>-1$ ($\nu=-1/2+i\tau$ for the case of conical functions) can be written in terms of the Gauss hypergeometric function as follows

$$P_\nu^{-\mu}(x)=\frac{1}{\Gamma(\mu+1)}\left|\frac{1-x}{1+x}\right|^{\mu/2}{}_2F_1\left(\begin{matrix}-\nu,\ \nu+1\\ \mu+1\end{matrix};\tfrac12-\tfrac12 x\right). \tag{2.1}$$

We adopt this definition for $x>-1$; this is the definition used in applications (both for $-1<x<1$ and $x>1$). The absolute value in the previous formula is showing that in fact we are dealing with two different functions in the complex plane (though trivially related). The functions defined in such a way satisfy the associated Legendre differential equation

$$(1-x^2)y''(x)-2xy'(x)+\left(\nu(\nu+1)-\mu^2/(1-z^2)\right)y(x)=0\,. \tag{2.2}$$

We can use this definition for real $x>-1$ except when $\mu\in\mathbb{Z}^-$, because for these values of μ the hypergeometric function in (2.1) is not defined. However, the reciprocal gamma function makes the right-hand side of (2.1) regular when $\mu\in\mathbb{Z}^-$. For $x\in(-1,1)$ this also follows from the relation

$$P_\nu^{\mu}(x)=\frac{\Gamma(\mu-\nu)\Gamma(\mu+\nu+1)}{\pi}\left[\sin(\pi\mu)\,P_\nu^{-\mu}(-x)-\sin(\pi\nu)\,P_\nu^{-\mu}(x)\right], \tag{2.3}$$

and we see that, when $m\in\mathbb{Z}$,

$$P_\nu^{m}(x)=-\frac{\Gamma(m-\nu)\Gamma(m+\nu+1)}{\pi}\sin(\pi\nu)\,P_\nu^{-m}(x). \tag{2.4}$$

This relation also holds for $x > 1$ [1, Eq. 8.2.5].

From these representations of the Legendre function, it is clear that when $\nu = -1/2 + i\tau$, $\tau \in \mathbb{R}$, the function $P_\nu^\mu(x)$ ($\mu \in \mathbb{R}$) remains real valued, as also does the defining differential equation and the rest of relations. For example, Eq. (2.4) reads:

$$P^m_{-\frac{1}{2}+i\tau}(x) = \cosh(\pi\tau)\frac{|\Gamma(m+1/2+i\tau)|^2}{\pi}P^{-m}_{-\frac{1}{2}+i\tau}(x). \tag{2.5}$$

We will use greek letters (μ) for denoting real values of the order and latin letters (m) for integer orders, expect otherwise specified.

3 Recurrence relation and continued fraction

Three-term recurrence relations are useful methods of computation when two starting values are available for starting the recursive process. Usually, the direction of application of the recursion can not be chosen arbitrarily, and the conditioning of the computation of a given solution fixes the direction. More explicitly, when the wanted function is recessive in a given recursion direction, one should use the opposite direction of recursion. As we discuss next, the recursion for conical functions admits recessive (also called minimal) solution, which indeed fixes the direction of stable recursion. First we study the case $x \in (-1, 1)$ and later the case $x > 1$ (the case $x = 1$ is trivial, and no minimal solution exists). The main tool is Perron's theorem [6, 15].

Conical functions $P^\mu_{-1/2+i\tau}(x)$ satisfy the following three-term recurrence relation when $x \in (-1, 1)$:

$$P^{\mu+1}_{-\frac{1}{2}+i\tau}(x) + \frac{2\mu x}{\sqrt{1-x^2}}P^{\mu}_{-\frac{1}{2}+i\tau}(x) - \left((\mu-\tfrac{1}{2})^2+\tau^2\right)P^{\mu-1}_{-\frac{1}{2}+i\tau}(x) = 0\,. \tag{3.1}$$

We start by studying the conditioning of recursion as $\mu \to -\infty$; for this, we write $y_\mu(x) = P^{-\mu}_{-1/2+i\tau}(x)$. Then we have

$$y_{\mu+1}(x) + b_\mu(x)y_\mu(x) + a_\mu(x)y_{\mu-1}(x) = 0\,, \tag{3.2}$$

and the behavior of the coefficients as $\mu \to +\infty$ is

$$\begin{aligned} &b_\mu(x) \sim b\mu^{-1}, \quad a_\mu(x) \sim a\mu^{-2}, \\ &b = 2x/\sqrt{1-x^2}, \quad a = -1. \end{aligned} \tag{3.3}$$

The application of Perron's theorem shows that the recurrence admits a minimal solution as $\mu \to +\infty$. The recurrence admits solutions with the asymptotic behavior

$$\frac{f_{\mu+1}}{f_\mu} \sim t_1 \mu^{-1}, \quad \frac{g_{\mu+1}}{g_\mu} \sim t_2 \mu^{-1}, \tag{3.4}$$

where t_i are the roots of the characteristic equation $t^2 + bt + a = 0$, namely

$$t_1 = \sqrt{\frac{1-x}{1+x}}, \quad t_2 = -\sqrt{\frac{1+x}{1-x}}. \tag{3.5}$$

The minimal solution is the one corresponding with the smallest $|t_i|$ (t_1 for $x > 0$ and t_2 for $x < 0$).

Next, we prove that $y_\mu(x) = P^{-\mu}_{-1/2+i\tau}(x)$ is minimal as $\mu \to +\infty$ when $0 < x < 1$ and dominant when $-1 < x < 0$. On the other hand, it is easy to check that the recurrence does not have minimal or dominant solutions when $x = 0$.

Indeed, from Eq. (2.1) we have

$$\frac{y_{\mu+1}}{y_\mu} \sim \frac{1}{\mu}\sqrt{\frac{1-x}{1+x}} \frac{{}_2F_1\left(\begin{matrix} -\nu,\ \nu+1 \\ \mu+2 \end{matrix}; \frac{1}{2} - \frac{1}{2}x\right)}{{}_2F_1\left(\begin{matrix} -\nu,\ \nu+1 \\ \mu+1 \end{matrix}; \frac{1}{2} - \frac{1}{2}x\right)}. \tag{3.6}$$

But, taking into account the asymptotic behavior of the Gauss $(0\,0+)$ recurrence [7], we have

$$\frac{y_{\mu+1}(x)}{y_\mu(x)} \sim \frac{1}{\mu}\sqrt{\frac{1-x}{1+x}}. \tag{3.7}$$

This shows that $y_\mu(x)$ is minimal as $\mu \to +\infty$ in $(0,1)$ and dominant in $(-1,0)$. Therefore, backward recursion for the computation of $y_\mu(x)$ from large positive values of μ is well-conditioned for $x \in (0,1)$, but only forward recursion (increasing μ) can be used for $x \in (-1,0)$.

For $x > 1$ the situation is very similar to the case $0 < x < 1$. In this case, we have the recurrence relation

$$P^{\mu+1}_{-\frac{1}{2}+i\tau}(x) + \frac{2\mu x}{\sqrt{x^2-1}} P^{\mu}_{-\frac{1}{2}+i\tau}(x) + \left((\mu - \tfrac{1}{2})^2 + \tau^2\right) P^{\mu-1}_{-\frac{1}{2}+i\tau}(x) = 0\,, \tag{3.8}$$

and proceeding as before we see that $y_\mu(x) = P^{-\mu}_{-1/2+i\tau}(x)$ satisfies a recurrence

$$y_{\mu+1}(x) + b_\mu(x) y_\mu(x) + a_\mu(x) y_{\mu-1}(x) = 0, \tag{3.9}$$

with

$$\begin{aligned} &b_\mu \sim b\mu^{-1}, \quad a_\mu \sim a\mu^{-2}, \\ &b = -2x/\sqrt{x^2-1}, \quad a = 1. \end{aligned} \tag{3.10}$$

The application of Perron's theorem shows that the recurrence admits a minimal solution as $\mu \to +\infty$. The recurrence admits solutions with the asymptotic behavior

$$\frac{f_{\mu+1}}{f_\mu} \sim t_1 \mu^{-1}, \quad \frac{g_{m+1}}{g_m} \sim t_2 \mu^{-1}, \tag{3.11}$$

with

$$t_1 = \sqrt{\frac{x-1}{x+1}}, \quad t_2 = \sqrt{\frac{x+1}{x-1}}. \tag{3.12}$$

The minimal solution corresponds to t_1 and, using again the known behavior of the $(0\,0\,+)$ Gauss recursion, we conclude that $y_\mu(x)$ is minimal when $x > 1$.

In the opposite direction of recursion, the situation is opposite and $P^{\mu}_{-1/2+i\tau}(x)$ is dominant when $x > 0$ (minimal when $x \in (-1,0)$) as $\mu \to \infty$; there is one important exception to this: when $\mu = m \in \mathbb{N}$. The fact that $P^{\mu}_{-1/2+i\tau}(x)$, $\mu \notin \mathbb{N}$, is dominant when $x > 0$ as $\mu \to \infty$ (minimal when $x < 0$) can be checked again using (2.1), except when $\mu = m \in \mathbb{N}$. On the other hand, using (2.4) we see at once that $P^{m}_{-1/2+i\tau}(x)$ is minimal when $x > 0$ (dominant when $x \in (-1,0)$) as m grows with positive integer m.

3.1 Continued fraction

When $x > 0$ (and backward recursion is stable) one needs two starting values with large $|m|$ for beginning the computation with the recurrence relation. Both values could be computed by means of the asymptotic expansions to be presented later, but in some cases, particularly near $x = 1$, it may be more efficient to compute the second value using a continued fraction.

Let us, for example, consider the case $m \in \mathbb{N}$. Then, considering the ratio $H_m = P^{m}_{-1/2+i\tau}(x)/P^{m-1}_{-1/2+i\tau}(x)$, which satisfies (see the recurrence relation (3.1))

$$H_m = \frac{(m-1/2)^2 + \tau^2}{\dfrac{2mx}{\sqrt{1-x^2}} + H_{m+1}}, \tag{3.13}$$

and the continued fraction that follows from this ratio, we see that we can compute $P^{m-1}_{-1/2+i\tau}(x)$ once $P^{m}_{-1/2+i\tau}(x)$ is known and H_m is computed numerically (see [6, chapter 6] for methods of computation).

The error in the computation of the continued fraction when N approximants are considered, is approximately given by (see [6, chapter 4])

$$\epsilon_k \simeq \left| \frac{g_M/f_M}{g_{M+k}/f_{M+k}} \right| , \tag{3.14}$$

where g_l and f_l are a dominant and the minimal solution of the three-term recurrence relation, respectively.

Using Perron's estimates for the ratios of functions, a rough estimate for the error is given by

$$\epsilon_N \sim \left(\frac{1-x}{1+x} \right)^N . \tag{3.15}$$

From (3.15), the number of approximants N which are needed in the computation of the continued fraction in order to obtain an accuracy ϵ, can be estimated as

$$N \sim \log(\epsilon)/\log \left| \frac{1-x}{1+x} \right| . \tag{3.16}$$

A test for this expression is shown in Fig. 1.

Similarly as before, when $x > 1$ an analogous representation for H_m is obtained from (3.8) and the number of approximants N which are needed in the computation of the continued fraction in order to obtain an accuracy ϵ, is again given by (3.16). Fig. 2 shows a comparison of this estimation with the actual values of N needed.

Notice that the continued fractions (both for $x > 1$ and $0 < x < 1$) converge very fast near $x = 1$, and that, therefore, around this value the continued fraction is an interesting method of computation.

As can be seen in Fig. 2, for large τ the error estimations don't work so well, not surprisingly because we are assuming that m is large with respect to the rest of parameters. Better estimations could be obtained from the uniform asymptotic expansions to be introduced in the next sections. In any case, the main conclusion is the same: that around $x = 1$ this is an effective method and that it can be used for computing one of the two starting values for starting recursions.

Figure 1: Test for (3.16). Comparison between the estimate for the number of approximants (crosses) of the continued fraction representation for (3.13) and the actual number of approximants (circles and diamonds), as a function of x (starting from $x = 0.4$) and for $m = 50$ and $\tau = 1.1, 50$ (circles and diamonds, respectively).

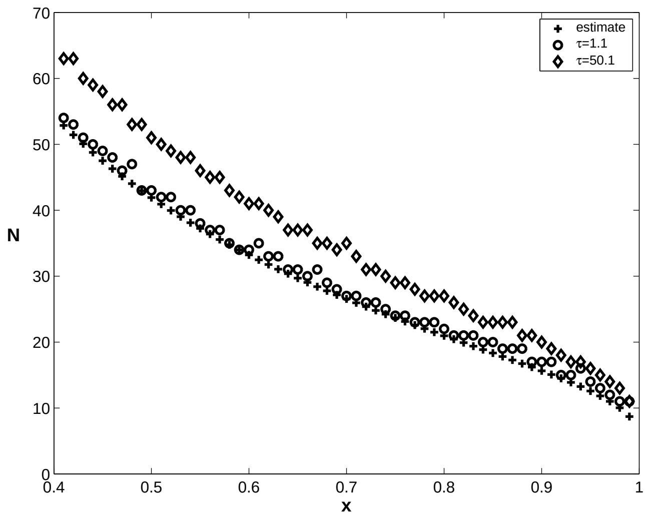
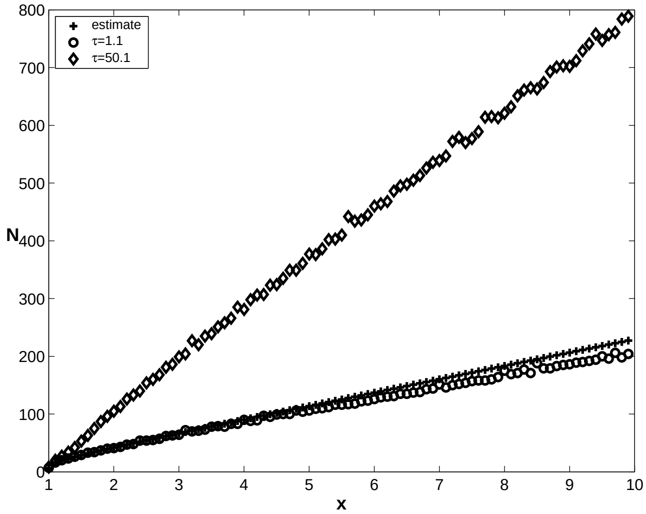


Figure 2: Test for (3.16) for $x > 1$. Comparison between the estimate for the number of approximants (crosses) of the continued fraction representation for (3.13) and the actual number of approximants (circles and diamonds), as a function of x and for $m = 50$ and $\tau = 1.1$ (circles) and $\tau = 50.1$ (diamonds).



4 Uniform asymptotic expansion for $x \in [-1,1]$

In this section we describe a powerful asymptotic expansion for $P^{-\mu}_{-1/2+i\tau}(x)$ that holds for large positive values of μ, and which is uniformly valid with respect to $\tau \ge 0$ and $x \in [-1,1]$. In §8, Appendix A, we give further details of this expansion, which reads

$$P^{-\mu}_{-\frac{1}{2}+i\tau}(x) \sim \sqrt{\frac{p}{x\mu}}\,\frac{\Gamma(\frac{1}{2}+\mu)\,(1-x^2)^{\mu/2}e^{-\mu\phi(t_0)}}{\Gamma(\mu+\frac{1}{2}-i\tau)\Gamma(\mu+\frac{1}{2}+i\tau)}\sum_{k=0}^{\infty}\frac{u_k(\beta,p)}{\mu^k}. \tag{4.1}$$

The quantities β, p and $\phi(t_0)$ are given by

$$\beta = \frac{\tau}{\mu}, \quad p = \frac{x}{\sqrt{1+\beta^2(1-x^2)}}, \tag{4.2}$$

and

$$\phi(t_0) = \ln\frac{x(p+1)}{p(\beta^2+1)} + \beta\arccos\frac{x(1-p\beta^2)}{p(1+\beta^2)}. \tag{4.3}$$

The first few coefficients of the expansion in (4.1) are

$$\begin{aligned} u_0(\beta,p) &= 1, \quad u_1(\beta,p) = -\frac{-\beta^2+5\beta^2p^3-3\beta^2p+3p}{24(\beta^2+1)},\\ u_2(\beta,p) &= \frac{1}{1152(\beta^2+1)^2}[385\beta^4p^6+462\beta^2(1-\beta^2)p^4-10\beta^4p^3\\ &\quad +(81\beta^4-522\beta^2+81)p^2+6\beta^2(\beta^2-1)p+\beta^4+72\beta^2-72]. \end{aligned} \tag{4.4}$$

As it is given, the asymptotic relation in (4.1) has no meaning at the points $x=\pm1$, because of the singularities of the conical function (and terms at the right-hand side of (4.1)) for these values of x. By using suitable scaling we can give representations that also hold at the endpoints (see for details §8.1).

For $x \sim -1$ the quantity $\phi(t_0)$ becomes singular and should be combined with powers of $(1+x)$. This gives for $-1 \le x \le 0$ the expansion

$$\begin{aligned} \left(\frac{1+x}{1-x}\right)^{\mu/2} P^{-\mu}_{-\frac{1}{2}+i\tau}(x) \;\sim\; & \sqrt{\frac{p}{x\mu}}\,\frac{\Gamma(\frac{1}{2}+\mu)e^{-\tau\arccos\frac{x(1-p\beta^2)}{p(1+\beta^2)}}}{\Gamma(\mu+\frac{1}{2}-i\tau)\Gamma(\mu+\frac{1}{2}+i\tau)}\times\\ & \left(\frac{x(1-p)}{p(1-x)}\right)^{\mu}\sum_{k=0}^{\infty}\frac{u_k(\beta,p)}{\mu^k}. \end{aligned} \tag{4.5}$$

For $x \sim 1$ the quantity $\phi(t_0)$ remains regular, and we can write for $0 \le x \le 1$:

$$\left(\frac{1+x}{1-x}\right)^{\mu/2} P^{-\mu}_{-\frac{1}{2}+i\tau}(x) \sim \sqrt{\frac{p}{x\mu}}\, \frac{\Gamma(\frac{1}{2}+\mu)(1+x)^{\mu} e^{-\mu\phi(t_0)}}{\Gamma(\mu+\frac{1}{2}-i\tau)\Gamma(\mu+\frac{1}{2}+i\tau)} \times \sum_{k=0}^{\infty} \frac{u_k(\beta,p)}{\mu^k}, \tag{4.6}$$

When μ is an integer ($\mu = m$) the product of the complex gamma functions in (4.1) , (4.5) and (4.6) can be written as

$$\Gamma(m+\tfrac{1}{2}-i\tau)\Gamma(m+\tfrac{1}{2}+i\tau) = \frac{\pi}{\cosh(\pi\tau)} \prod_{n=1}^{m} \left((m-n+\tfrac{1}{2})^2+\tau^2\right). \tag{4.7}$$

4.1 Numerical tests of the expansion for $x \in [-1, 1]$

In order to test the range of validity of the expansion in (4.1), we first compare it with the values obtained from Eq. (2.1) by using Maple, computing the function with as many digits as needed (more digits as τ becomes larger).

Fig. 3 shows, as a function of x, the minimum value of m for which the use of (4.1) allows to get single precision (10^{-8}) in the computation of $P^{-m}_{-1/2+i\tau}(x)$. We have used expansion (4.1) with terms $0 \le k \le 7$. Fig. 4 shows the corresponding results for double precision (10^{-16}).

A second test, independent of Maple computations, has been considered. It consists in testing that the values computed from asymptotics are consistent with the three term recurrence relation (3.1). For this, when backward recursion is stable ($x > 0$), we compute $P^{-m-1}_{-1/2+i\tau}(x)$ and $P^{-m}_{-1/2+i\tau}(x)$ from asymptotics and obtain $P^{-m+1}_{-1/2+i\tau}(x)$ by applying (3.1). This value is tested against the direct computation of $P^{-m+1}_{-1/2+i\tau}(x)$ from the asymptotic expansion (4.1). For $x < 0$, we proceed similarly but in the opposite direction of recursion ($P^{-m-1}_{-1/2+i\tau}(x)$ is obtained from $P^{-m}_{-1/2+i\tau}(x)$ and $P^{-m+1}_{-1/2+i\tau}(x)$). Fig. 5 illustrates this test as a function of m.

The variation with respect to x of the accuracy of the asymptotic expansion is illustrated in Fig. 6. The figure shows the relative errors obtained by comparing the scaled forms of the expansion (4.5), (4.6) against Maple. The values of the parameters τ and m are $10.1, 100.1$ and 40, respectively.

Figure 3: Minimum values of m for which the use of (4.1) allows to get a precision better than 10^{-8} in the computation of $P^{-m}_{-1/2+i\tau}(x)$.

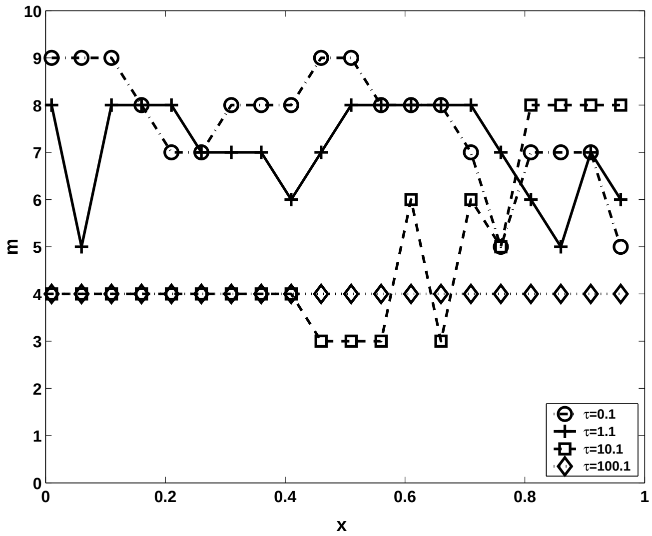


Figure 4: Minimum values of m for which the use of (4.1) allows to get a precision better than 10^{-16} in the computation of $P^{-m}_{-1/2+i\tau}(x)$.

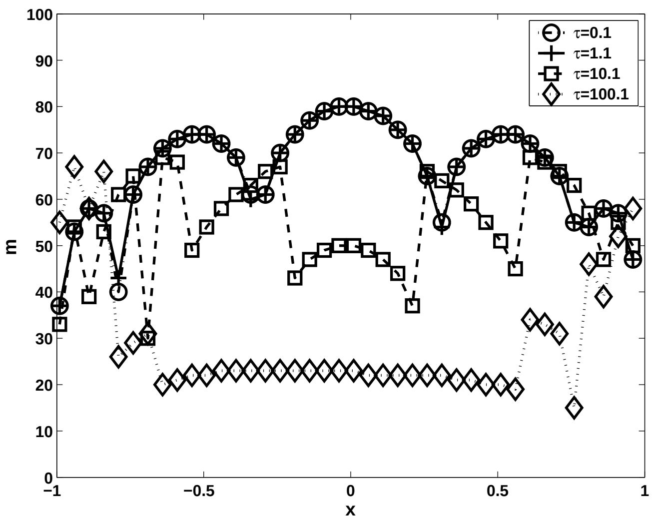


Figure 5: Relative errors in the computation of $P^{-m+1}_{-1/2+i\tau}(x)$ obtained by comparing (4.1) and (3.1). In this figure $\tau = 100.1$ and $x = 0.4$.

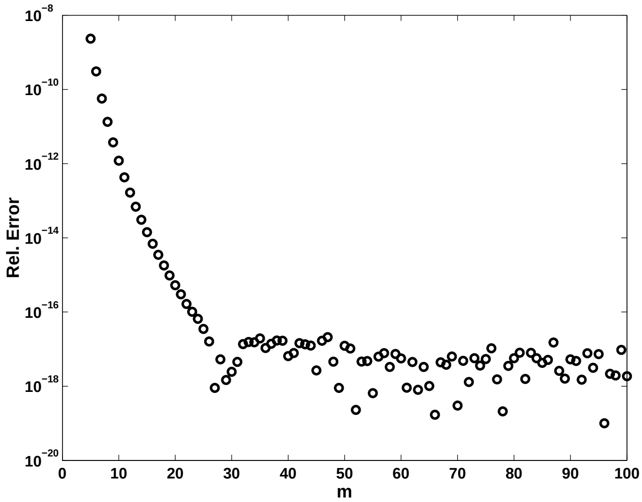
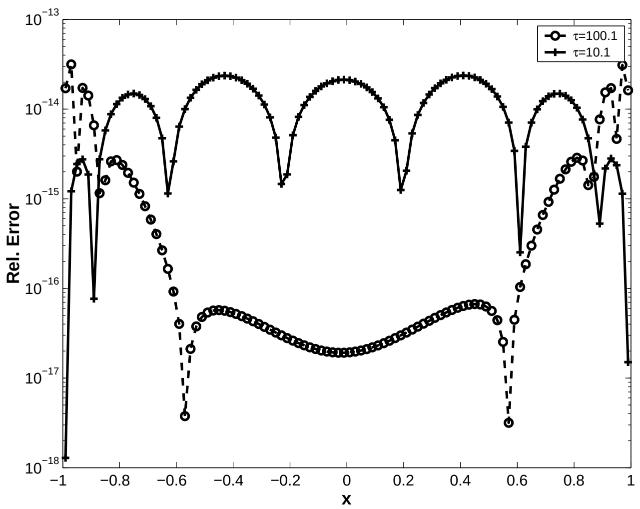


Figure 6: Relative errors in the computation of $P^{-m+1}_{-1/2+i\tau}(x)$ obtained by comparing (4.5), (4.6) against Maple. In this figure $\tau = 10.1, 100.1$ and $m = 40$.



5 Quadrature methods for $x \in [-1, 1]$

As explained extensively in [6, Ch. 5], knowing details of saddle point contours of integrals defining special functions, we can use this information for constructing efficient quadrature methods for evaluating these integrals. We derive an integral that can be used for all non-negative values of the parameters μ and τ (large or small) and for all $x \in [-1, 1]$. For example, numerical quadrature can be used to compute starting values of the recursion when $x \in (-1, 0]$; see §3.

We use the integral in (8.2) that is used for obtaining the asymptotic expansion of the previous section. This integral has a saddle point at t_0 given in (8.8), and we integrate along the horizontal line through t_0. That is, we write $t = s + is_0$, where $is_0 = t_0$, and denote the integral in (8.2) with I. We obtain

$$I = e^{-\mu\phi(t_0)} \int_{-\infty}^{\infty} e^{-\mu\psi(s)} \frac{ds}{\sqrt{x + \cosh(s + s_0)}}, \tag{5.1}$$

where $\phi(t_0)$ is given in (4.3) and

$$\psi(s) = \ln\left(1 + \frac{2\cos(s_0)\sinh^2(s/2)}{x + \cos(s_0)} + i\frac{\sin(s_0)\sinh(s)}{x + \cos(s_0)}\right) - i\beta s, \tag{5.2}$$

where

$$\cos(s_0) = \frac{x(1 - p\beta^2)}{p(1 + \beta^2)}, \quad \sin(s_0) = \frac{\beta x(p + 1)}{p(1 + \beta^2)}, \tag{5.3}$$

and β and p are as in (4.2).

We write $\psi(s)$ in the form $\psi(s) = \psi_r(s) + i\psi_i(s)$, where

$$\begin{aligned}
\psi_r(s) &= \tfrac{1}{2}\ln\left(1 + \frac{4(1 + \beta^2)}{1 + p}\sigma^2 + \frac{4(1 + \beta^2)(1 + p^2\beta^2)}{(1 + p)^2}\sigma^4\right), \\
\psi_i(s) &= \arctan\frac{\beta(1 + p)\sinh s}{1 + p + (1 - p\beta^2)\sinh^2(s/2)} - \beta s,
\end{aligned} \tag{5.4}$$

where $\sigma = \sinh(\frac{1}{2}s)$. We have, as $s \to 0$,

$$\begin{aligned}
\psi_r(s) &= \frac{1 + \beta^2}{2(1 + p)}s^2 + \frac{(1 + \beta^2)(p - 2 + 3\beta^2(p^2 - 2))}{24(1 + p)^2}s^4 + \mathcal{O}(s^6), \\
\psi_i(s) &= \frac{\beta(1 + \beta^2)(p - 2)}{6(1 + p)}s^3 + \mathcal{O}(s^5).
\end{aligned} \tag{5.5}$$

After these preparations, we obtain the desired integral representation

$$I = 2e^{-\mu\phi(t_0)}\sqrt{\frac{p(1+\beta^2)}{x(p+1)}}\int_0^\infty e^{-(\mu+\frac12)\psi_r(s)}\cos((\mu+\tfrac12)\psi_i(s))\,ds. \tag{5.6}$$

An efficient quadrature method can be based on the trapezoidal rule. For details we refer to [6, Ch. 5].

For $x \sim -1$ we have $p \sim -1$, and the representation should be modified. As explained in §4, see (4.5), the quantity $\phi(t_0)$ also becomes singular, but a suitable scaling of the conical function may take care of the factor $\exp(-\mu\phi(t_0))$. The factor $\sqrt{p+1}$ in (5.6) can be handled, for example, by introducing the new variable of integration $w = \sinh(s/2)/\sqrt{1+p}$. Then

$$\frac{ds}{dw} = 2\sqrt{\frac{1+p}{1+(1+p)w^2}}, \tag{5.7}$$

and in the new integral we can use $x \sim -1$ without any problem. The new integral is not fast convergent, but acceleration methods can be used as described in [6, § 5.4.2].

6 Uniform asymptotic expansion for $x \geq 1$

To distinguish between the previous cases with $x \in [-1, 1]$, we now write $z = x$, $z \geq 1$. This is the area where zeros occur, and for describing the transition between the monotonic behavior and oscillatory behavior, an expansion in terms of elementary functions is no longer adequate. We have used a representation in terms of the modified Bessel function $K_{i\tau}(\mu\zeta)$, which reads (for details we refer to §9, Appendix B):

$$\begin{aligned} P^{-\mu}_{-\frac12+i\tau}(z) = {} & \frac{2\Gamma(\frac12+\mu)\,(z^2-1)^{\mu/2}\,e^{-\mu\lambda}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)}\,\Phi(\zeta)\times \\ & \left[A_\mu(\beta,\zeta)K_{i\tau}(\mu\zeta) - B_\mu(\beta,\zeta)K'_{i\tau}(\mu\zeta)\right], \end{aligned} \tag{6.1}$$

where

$$\lambda = \frac12\left(\ln\frac{z^2-1}{\beta^2+1} + \beta\arccos\frac{1-\beta^2}{1+\beta^2}\right), \tag{6.2}$$

$$\beta = \frac{\tau}{\mu}, \quad \Phi(\zeta) = \left(\frac{\zeta^2-\beta^2}{1+\beta^2(1-z^2)}\right)^{\frac14}, \tag{6.3}$$

and the functions $A_\mu(\beta,\zeta)$ and $B_\mu(\beta,\zeta)$ have the expansions

$$A_\mu(\beta,\zeta) \sim \sum_{n=0}^{\infty} \frac{A_n(\beta,\zeta)}{\mu^n}, \quad B_\mu(\beta,\zeta) \sim \sum_{n=0}^{\infty} \frac{B_n(\beta,\zeta)}{\mu^n}. \tag{6.4}$$

For describing the parameter ζ and the coefficients of the expansions we use two different $z-$intervals. Let

$$z_c = \frac{\sqrt{1+\beta^2}}{\beta}. \tag{6.5}$$

6.1 The monotonic case: $1 \le z \le z_c$

In this case the quantity $\zeta \ge \beta$ is given by the implicit equation

$$2\left[\sqrt{\zeta^2-\beta^2} - \beta \arccos(\beta/\zeta)\right] = \ln \frac{p+1}{p-1} - \beta \arccos \frac{\beta^2 p^2 - 1}{\beta^2 p^2 + 1}, \tag{6.6}$$

where p is given by

$$p = \frac{z}{\sqrt{1+\beta^2(1-z^2)}}. \tag{6.7}$$

For the numerical inversion of this equation, that is, computing ζ when z is given, we refer to §6.1.1 below.

The first few coefficients $A_n(\beta,\zeta), B_n(\beta,\zeta)$ in (6.4) are

$$A_0(\beta,\zeta) = 1, \quad B_0(\beta,\zeta) = 0, \quad A_1(\beta,\zeta) = \frac{\beta^2}{24(1+\beta^2)}, \tag{6.8}$$

$$B_1(\beta,\zeta) = -\frac{(5\beta^2(W^3p^3 - 1 - \beta^2) + 3W^2(Wp(1-\beta^2) - 1 - \beta^2)\zeta}{24W^4(1+\beta^2)}, \tag{6.9}$$

where p is given in (6.7) and

$$W = \sqrt{\zeta^2 - \beta^2}. \tag{6.10}$$

Observe that the function $\Phi(\zeta)$ given in (6.3) can be written as $\Phi(\zeta) = \sqrt{pW/z}$. This functions is analytic at the point $\zeta = \beta$, and can be defined across this point by changing the sign under the square roots in (6.7) and (6.10) simultaneously.

6.1.1 Numerical inversion of the implicit equation (6.6)

The left-hand side in (6.6) vanishes at $\zeta = \beta$ and is a strictly increasing function of $\zeta \in [\beta, \infty)$; the derivative with respect to ζ is $2\sqrt{1-\beta^2/\zeta^2}$. The right hand side becomes $+\infty$ as $z \downarrow 1$ and vanishes at $z = z_c$; it is strictly decreasing function of $z \in [1, z_c]$, the derivative with respect to z being $-2\sqrt{1+\beta^2(1-z^2)}/(z^2-1)$. It follows that from (6.6) a unique value ζ can be obtained when values of z and β are given. Conversely, we can obtain a unique value z when values of ζ and β are given.

The derivatives are similar to those used in the Liouville transformation in [2, Eq. (4.5)], after using in our analysis $\zeta = \sqrt{\eta}$ and $z^2 = 1 + 1/\xi$.

An explicit analytical inversion of equation (6.6) is not possible, but one can find initial approximations which guarantee convergence of Newton's method. We rewrite the equation as

$$\sqrt{\gamma^2-1} - \arccos(1/\gamma) = \Omega \tag{6.11}$$

$\gamma = \zeta/\beta$, and $\Omega = f(p,\beta)/(2\beta)$ and $f(p,\beta)$ is the right-hand side of Eq. (6.6). For large γ, we expand the left hand side and get, after inverting the asymptotic series (see [6, Chapter 7]), the approximation

$$\gamma \sim \Omega + \frac{\pi}{2} - \frac{1}{2\Omega} + \frac{\pi}{4\Omega^2} + \mathcal{O}(\Omega^{-3}). \tag{6.12}$$

This approximation is, of course, better as Ω is large, but can be used as starting value for $\Omega \geq 1$.

For smaller values of Ω, which leads to smaller values of γ, we can expand the left hand side of (6.11) in powers of $\gamma - 1$. With the leading term of the expansion, we get

$$\gamma \sim 1 + \left(\frac{3\Omega}{2\sqrt{2}}\right)^{2/3}. \tag{6.13}$$

This approximation gives a convenient starting value for Newton's method when $0 \leq \Omega \leq 1$.

6.2 The oscillatory case: $z \geq z_c$

In this case the quantity $\zeta \in [0, \beta]$ is given by the implicit equation

$$2\left[\sqrt{\beta^2-\zeta^2} - \beta \operatorname{arccosh}(\beta/\zeta)\right] = 2 \operatorname{arccot} q - \beta \ln \frac{\beta q + 1}{\beta q - 1}, \tag{6.14}$$

where q is given by

$$q = \frac{z}{\sqrt{\beta^2(z^2-1)-1}}. \tag{6.15}$$

The coefficients $A_0(\beta,\zeta), B_0(\beta,\zeta), A_1(\beta,\zeta)$ are as in (6.8), whereas the coefficient $B_1(\beta,\zeta)$ is given by

$$B_1(\beta,\zeta) = -\frac{(5\beta^2(V^3q^3-1-\beta^2)+3V^2(Vq(1-\beta^2)-1-\beta^2)\zeta}{24V^4(1+\beta^2)}, \quad (6.16)$$

where $V=\sqrt{\beta^2-\zeta^2}$. In fact this $B_1(\beta,\zeta)$ is the same as the one in (6.9), with proper interpretation of the square roots in W, p and V, q. See the remark at the end of §6.1. The function $\Phi(\zeta)$ given in (6.3) can also be written as $\Phi(\zeta)=\sqrt{qV/z}$. More details on the relation between p and W for $z \sim z_c$ (that is, for small values of W) will be given in §9.5..

6.2.1 Numerical inversion of the implicit equation (6.14)

The derivative with respect to ζ of the left-hand side of (6.14) is $2\sqrt{\beta^2/\zeta^2-1}$, and that of the right-hand side equals $-2\sqrt{\beta^2(z^2-1)-1}/(z^2-1)$. It follows that (6.14) defines a unique value ζ when values of z and β are given, and conversely, we can obtain a unique value z when values of ζ and β are given.

Again, Eq. (6.14) cannot be explicitly inverted, but accurate enough values for the Newton method can be determined. Writing the equation as

$$\sqrt{\gamma^2-1} - \operatorname{arccosh}(1/\gamma) = \Lambda \quad (6.17)$$

$\gamma=\zeta/\beta$, and $\Lambda = \dfrac{g(p,\beta)}{2\beta}$ and $g(p,\beta)$ is the right-hand side of Eq. (6.14) we get, by expanding the left-hand side of the equation in powers of $\gamma-1$ that for values of γ close to 1 the following approximation can be used

$$\gamma \simeq 1 - \left(\frac{-3\Lambda}{2\sqrt{2}}\right)^{2/3} \quad (6.18)$$

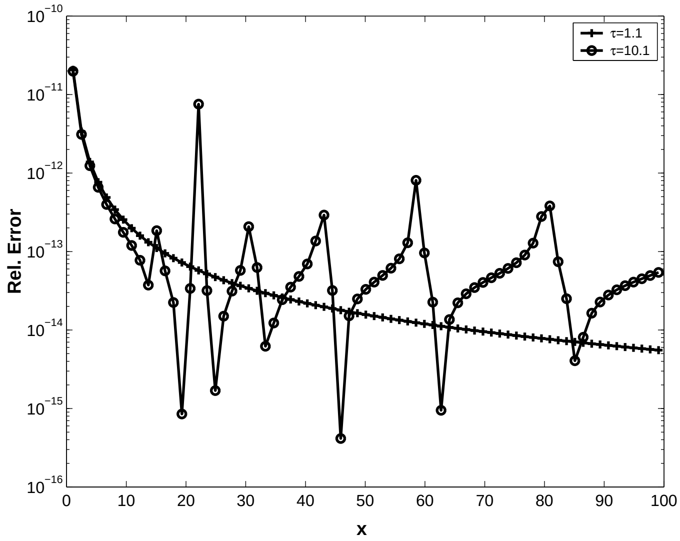
The approximation is sufficient for $-0.5 \le \Lambda \le 0$ while for $\Lambda < -0.5$ (and smaller γ), it is better to expand the left-hand side and invert the expansion. The following approximation can then be obtained after two resubstitutions

$$\gamma \sim 2e^{\Lambda-1} \quad (6.19)$$

6.3 Numerical test of the expansions for $x > 1$

Fig. 7 shows, as a function of x and for two values of τ, the relative error in the computation of $P^{-m}_{-1/2+i\tau}(x)$ by using (6.1) and Maple. We have used

Figure 7: Relative errors in the computation of $P^{-m+1}_{-1/2+i\tau}(x)$ for $m = 100$ and $\tau = 1.1, 10.1$.



expansion (6.4) with terms $0 \le k \le 3$. The oscillations in the relative error for $\tau = 10.1$ can be explained by the loss of relative accuracy near the zeros of the function. Indeed, conical functions oscillate for $x > 1$ (and have an infinite number of zeros).

As in the case $x \in [-1, 1]$, the asymptotic expansion has been also tested by checking the three term recurrence relation (3.1). Fig. 8 shows, as a function of x and for the same two values of τ used in Fig. 7, the relative error in the computation of $P^{-m+1}_{-1/2+i\tau}(x)$ by using (6.1) and (3.1). The results are consistent with those of Fig. 7.

Finally, Fig. 9 shows, as a function of x, the minimum value of m for which the use of (6.1) allows to get single precision (10^{-8}) in the computation of $P^{-m}_{-1/2+i\tau}(x)$ for $\tau = 1.1, 10.1$.

7 Computational scheme

We have that $y_\mu(x) = P^{-\mu}_{-1/2+i\tau}(x)$ is minimal as $\mu \to +\infty$ when $x > 0$ and dominant when $x < 0$; at $x = 0$ it is neither minimal nor dominant and at $x = 1$ the computation through recurrence is undefined (and unnecessary).

Figure 8: Relative errors in the computation of $P^{-m+1}_{-1/2+i\tau}(x)$ by using (6.1) and (3.1) for $m = 100$ and $\tau = 1.1, 10.1$.

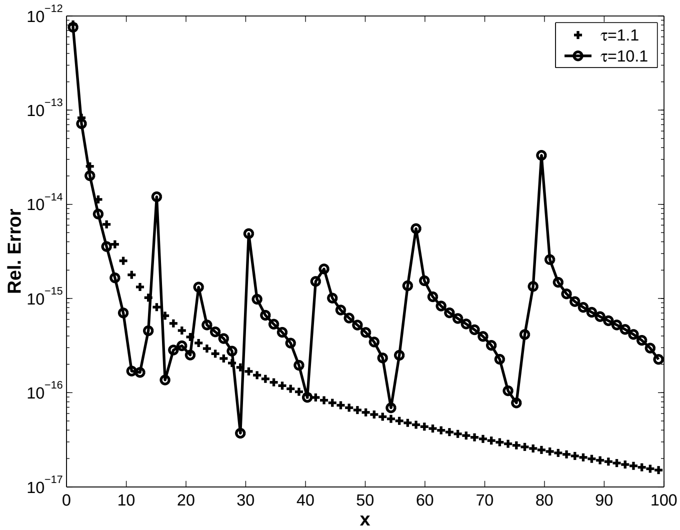
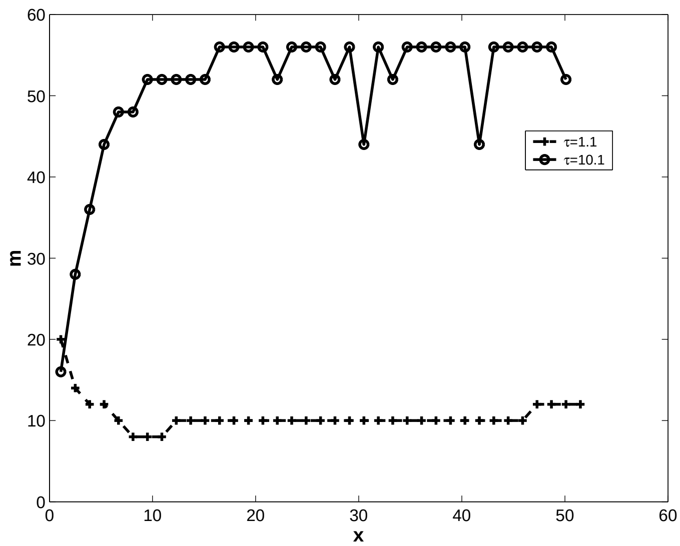


Figure 9: Minimum values of m for which the use of (6.1) allows to get a precision better than 10^{-8} in the computation of $P^{-m}_{-1/2+i\tau}(x)$.



Therefore, for computing $P^{-\mu}_{-1/2+i\tau}(x)$, $x > 0$, $\mu \ge 0$, recursion starting from large μ and decreasing μ is possible. And for $P^{m}_{-1/2+i\tau}(x)$, $m \in \mathbb{N}$, the relation (2.5) can be used or, alternatively, backward recursion is possible starting with large positive m. The starting values for these recurrences are provided by the uniform asymptotic expansions discussed in sections 4 and 6; the values of m which can be used for starting the recursion can be extracted from the information of the type described Figures 3, 4 and 9.

When $x \in (-1, 0)$ the recurrent scheme changes and conical functions $P^{\mu}_{-1/2+i\tau}(x)$ with $\mu \le 0$ or $\mu = m \in \mathbb{N}$ should be computed in the direction of increasing $|\mu|$. Fortunately, methods for computing conical function for small μ are already available. And this, together with backward recursion and asymptotics for positive x, gives a complete scheme of stable computation.

For negative x and $\mu = m \in \mathbb{N}$ one can start from the values $P^{0}_{-1/2+i\tau}(x)$ and $P^{1}_{-1/2+i\tau}(x)$ (which are computed in [8]) and use the recurrence (3.1) for computing for $m > 1$; analogously, one can start with two negative values of the order of small magnitude and use (3.9) for computing values $P^{-\mu}_{-1/2+i\tau}(x)$ for large μ. For computing $P^{m}_{-1/2+i\tau}(x)$ for small m, Kölbig [8] proposed using power series in terms of Gauss functions. One possibility is to use (2.1), which should be computed by other means close to $x = -1$ in order to avoid bad behavior of the Gauss series (with argument $z \simeq 1$). Kölbig suggested rational approximations for the Gauss function. Another possibility is to compute the initial values for the recurrence by means of integral representations; see § 5. The connection formula (2.3) is also useful for computations when μ is not an integer and x is close to -1.

In summary, the stable scheme of computation for positive integer or negative real orders μ consists in:

1. If $x > 0$, compute $P^{\mu}_{-1/2+i\tau}(x)$ for large values of $|\mu|$ and use recursion in the direction of decreasing $|\mu|$.
2. If $x < 0$, compute $P^{\mu}_{-1/2+i\tau}(x)$ for small values of $|\mu|$ and use recursion in the direction of increasing $|m|$.

8 Appendix A: Details of the uniform asymptotic expansion for $x \in [-1, 1]$

We give the details of the asymptotic expansion (4.1). Dunster [2, p. 326] has given an expansion for $x \in [0, 1]$ by using the differential equation of the conical functions. We derive a similar expansion for $P^{-\mu}_{\nu}(x)$ by using an integral representation, and conclude that the expansion is valid for $x \in [-1,1]$, after suitable scaling of the conical function.

Our starting point is the integral representation

$$P_\nu^{-\mu}(x) = \frac{\sqrt{2/\pi}\,\Gamma(\frac12+\mu)\,(1-x^2)^{\mu/2}}{\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_0^\infty (x+\cosh t)^{-\mu-\frac12}\,\cos(\tau t)\,dt, \tag{8.1}$$

valid for $x \in (-1,1)$ and $\mu > -\frac12$. This representation is given in [9, p. 188]. See also [10, p. 35].

The integrand is even, and we have

$$P_\nu^{-\mu}(x) = \frac{\Gamma(\frac12+\mu)\,(1-x^2)^{\mu/2}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_{-\infty}^\infty (x+\cosh t)^{-\mu-\frac12}\,e^{i\tau t}\,dt, \tag{8.2}$$

which we write in the form

$$P_\nu^{-\mu}(x) = \frac{\Gamma(\frac12+\mu)\,(1-x^2)^{\mu/2}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_{-\infty}^\infty e^{-\mu\phi(t)}\frac{dt}{\sqrt{x+\cosh t}}, \tag{8.3}$$

where

$$\phi(t) = \ln(x+\cosh t) - i\beta t, \quad \beta = \frac{\tau}{\mu}. \tag{8.4}$$

The integral has a saddle point where $\phi'(t) = 0$. That is, we have to solve the equation

$$\frac{\sinh t}{x+\cosh t} - i\beta = 0. \tag{8.5}$$

By using the exponential representation of the hyperbolic functions, we obtain for the solution t_0 the relation

$$e^{t_0} = \frac{i\sqrt{1+\beta^2(1-x^2)} - \beta x}{\beta + i}. \tag{8.6}$$

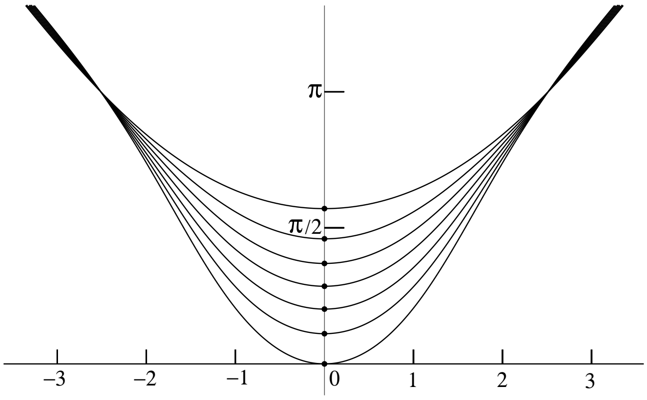
We have taken the + sign for the square root because we need the solution that gives $t_0 \sim 0$ if $\beta \to 0$. We write this in the form

$$e^{t_0} = x\frac{1-p\beta^2+i\beta(1+p)}{p(\beta^2+1)}, \quad p = \frac{x}{\sqrt{1+\beta^2(1-x^2)}}. \tag{8.7}$$

We have $-1 \le p \le 1$ for $-1 \le x \le 1$. The right-hand side of the first equation in (8.7) has absolute value equal to unity. Hence, t_0 is equal to the phase of that complex number and we have

$$t_0 = i\arctan\frac{\beta(1+p)}{1-p\beta^2} = i\arccos\frac{x(1-p\beta^2)}{p(1+\beta^2)}, \tag{8.8}$$

Figure 10: Saddle point contours governed by the equation (8.10) for $\beta = 1.25$ and $x = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$. The contour through the origin is for $x = -1$, the highest contour is for $x = 1$.



where we assume (for the arctan) that $1-p\beta^2 \geq 0$, otherwise we add π to the arctan. We prefer the notation in terms of the arccos function because the standard definition of the arctan function does not give the phase outside the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

There are more saddle points, all on the imaginary axis, but t_0 is the relevant saddle. There are also singularities on the imaginary axis, where $x + \cosh t = 0$, that is, at $t_s = i \arccos(-x)$ (and at other place, but t_s is the relevant singularity). We have

$$0 < \Im t_0 < \Im t_s, \tag{8.9}$$

for all $x \in (-1, 1)$ and $\beta \geq 0$.

We can shift the path of integration in (8.3) upwards, through the saddle at t_0, and deform the path along the saddle point contour through t_0 defined by

$$\Im \phi(t) = \Im \phi(t_0), \tag{8.10}$$

where $\Im \phi(t_0) = 0$. See Figure 10.

The quantity $\phi(t_0)$ is given in (4.3); furthermore we have

$$\phi''(t_0) = \frac{\beta^2 + 1}{p + 1}. \tag{8.11}$$

We transform

$$\phi(t) - \phi(t_0) = \tfrac{1}{2}\phi''(t_0) w^2, \tag{8.12}$$

and we assume that $\operatorname{sign}\Re(t-t_0)=\operatorname{sign}(w)$ when t is on the saddle point contour and the corresponding w on the real axis. This gives for the integral in (8.3)

$$\int_{-\infty}^{\infty} e^{-\mu\phi(t)}\frac{dt}{\sqrt{x+\cosh t}} = e^{-\mu\phi(t_0)}\int_{-\infty}^{\infty} e^{-\frac12\mu\phi''(t_0)w^2} f(w)\,dw, \tag{8.13}$$

where

$$f(w)=\frac{dt}{dw}\frac{1}{\sqrt{x+\cosh t}},\quad \frac{dt}{dw}=\phi''(t_0)\frac{w}{\phi'(t)}. \tag{8.14}$$

We expand

$$f(w)=\sum_{k=0}^{\infty} f_k w^k \tag{8.15}$$

and substitute this in (8.13). This gives the asymptotic expansion

$$\int_{-\infty}^{\infty} e^{-\mu\phi(t)}\frac{dt}{\sqrt{x+\cosh t}} \sim e^{-\mu\phi(t_0)}\sum_{k=0}^{\infty} f_{2k}\Gamma(k+\tfrac12)\left(\tfrac12\mu\phi''(t_0)\right)^{-k-\frac12}. \tag{8.16}$$

This can be written in the form

$$\int_{-\infty}^{\infty} e^{-\mu\phi(t)}\frac{dt}{\sqrt{x+\cosh t}} \sim \sqrt{\frac{2\pi p}{x\mu}}e^{-\mu\phi(t_0)}\sum_{k=0}^{\infty}\frac{u_k(\beta,p)}{\mu^k}, \tag{8.17}$$

where p/x is well defined when $x=0$, see (8.7), and

$$u_k(\beta,p)=\frac{2^k\,(\frac12)_k}{\phi''(t_0)^k}\,\frac{f_{2k}}{f_0},\quad k=0,1,2,\ldots\,. \tag{8.18}$$

Finally, by using (8.3), we obtain the expansion in (4.1) with the first coefficients given in (4.4).

8.1 Interpretation of the expansion at $x=\pm1$

The expansion in (4.1) remains valid at the endpoints $x=\pm1$, after properly scaling with the powers of $1-x^2$. For $x=-1$ we obtain from (2.1)

$$\lim_{x\downarrow -1}(1+x)^{\mu/2}P_\nu^{-\mu}(x)=\frac{2^{\mu/2}\,\Gamma(\mu)}{\Gamma(\mu+\frac12-i\tau)\Gamma(\mu+\frac12+i\tau)}, \tag{8.19}$$

where we have used

$${}_2F_1\left({a,\ b \atop c};1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \Re(c-a-b)>0. \tag{8.20}$$

Taking the same limit in (4.1) and comparing the two results, we obtain

$$\frac{\sqrt{\mu}\,\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})} \sim \sum_{k=0}^{\infty} \frac{u_k(\beta,-1)}{\mu^k}, \tag{8.21}$$

where the first few coefficients are

$$u_0(\beta,-1)=1, \quad u_1(\beta,-1)=\tfrac{1}{8}, \quad u_2(\beta,-1)=\tfrac{1}{128}. \tag{8.22}$$

These terms corresponds with the expansion first terms in the expansion of the ratio of gamma functions given in [1, Eq. 6.1.47].

Next, again from (2.1),

$$\lim_{x\uparrow 1}(1-x)^{-\mu/2}P_{\nu}^{-\mu}(x) = \frac{2^{-\mu/2}}{\Gamma(1+\mu)}, \tag{8.23}$$

and comparing the results of a similar limit in (4.1), we obtain

$$B(\mu+\tfrac{1}{2}-i\tau,\mu+\tfrac{1}{2}+i\tau) \sim \sqrt{\pi/\mu}\,2^{-2\mu}(1+\beta^2)^{\mu}e^{-2\tau\arctan\beta}\sum_{k=0}^{\infty}\frac{u_k(\beta,1)}{\mu^k}, \tag{8.24}$$

where $B(p,q)$ is the beta integral

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}=\int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \quad \Re p,q>0. \tag{8.25}$$

By using standard saddle point methods the expansion in (8.24) can be obtained directly from this integral representation with $p=\mu+\frac{1}{2}-i\tau$, $q=\mu+\frac{1}{2}+i\tau$.

9 Appendix B: Details of the uniform asymptotic expansion for $z\in[1,\infty)$

We give details of the asymptotic expansion (6.1) that holds for large values of μ, uniformly with respect to $z\in[1,\infty)$, and $\tau\ge 0$. A similar expansion has been given by Dunster [2, p. 325]. His method is based on the differential equation for the conical functions, and we are using an integral representation. This gives a more direct procedure for obtaining the coefficients in (6.4). A similar approach has been used in [14, §5].

We start with the same integral as in the previous section, see (8.3),

$$P^{-\mu}_{-\frac12+i\tau}(z)=\frac{\Gamma(\frac12+\mu)\,(z^2-1)^{\mu/2}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)}\int_{-\infty}^{\infty}e^{-\mu\phi(t)}\frac{dt}{\sqrt{z+\cosh t}},\qquad(9.1)$$

where

$$\phi(t)=\ln(z+\cosh t)-i\beta t,\quad \beta=\frac{\tau}{\mu},\quad \nu=-\tfrac12+i\tau,\qquad(9.2)$$

First we observe that the expansion given in (4.1) remains valid for $1\le z\le z_c-\delta$, where δ is a small positive number, and z_c is given in (6.5). We only have to replace $(1-x^2)^{\mu/2}$ with $(z^2-1)^{\mu/2}$ and x by z, also in the quantity p defined in (8.7). The equation for the saddle points

$$\phi'(t)=\frac{\sinh t}{z+\cosh t}-i\beta=0\qquad(9.3)$$

has two coalescing solutions when $1+\beta^2(1-z^2)=0$, that is, when $z=z_c$, which also become important in §4, when we would have considered $x>1$. The value z_c becomes large as $\beta=\tau/\mu$ becomes small. So, when $\tau\ll\mu$ we can still use the expansion (4.1) for a large $z-$interval, but the expansion becomes invalid when z approaches z_c.

9.1 The saddle points

As remarked above, when we repeat the saddle point analysis of §4, we observe that equation (9.3) with $z\ge1$ has two saddle points $t_\pm$ that coincide when $z=z_c$, where z_c is given in (6.5). In this section we allow $z\sim z_c$, however in the analysis we consider two cases: $1\le z\le z_c$ and $z_c\le z$.

9.1.1 The monotonic case: $1\le z\le z_c$

We have for the two saddle points the relations

$$e^{t_+}=z\frac{1-p\beta^2+i\beta(1+p)}{p(\beta^2+1)},\quad e^{t_-}=z\frac{1+p\beta^2+i\beta(1-p)}{-p(\beta^2+1)},\qquad(9.4)$$

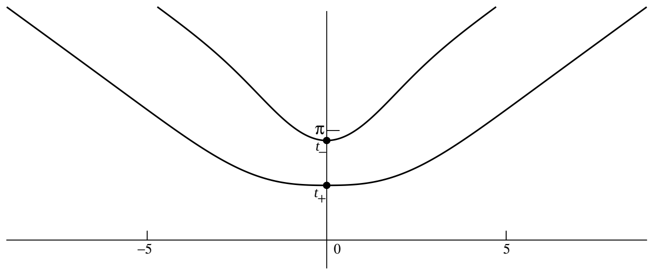
where p is given in (6.7).

Again the right-hand sides have modulus equal to unity, and we obtain

$$t_+=i\arccos\frac{z(1-p\beta^2)}{p(1+\beta^2)},\quad t_-=i\arccos\frac{-z(1+p\beta^2)}{p(1+\beta^2)}.\qquad(9.5)$$

For $\beta=0$ we have $t_+=0$ and $t_-=i\pi$. For $z=1$ we have $t_+=i\arccos(1-\beta^2)/(1+\beta^2)$ and again $t_-=i\pi$. This value of t_- follows correctly from the

Figure 11: Saddle point contours through the saddle points given in (9.5) for $\beta = \frac{3}{4}$ and $z = \frac{4}{3}$.



second equation in (9.4), but it is in fact not a correct saddle point. Namely, for $z = 1$ equation (9.3) becomes $\tanh \frac{1}{2}t = i\beta$, with solution

$$t_+ = 2i \arctan \beta = i \arctan \frac{2\beta}{1-\beta^2} = i \arccos \frac{(1-\beta^2)}{(1+\beta^2)}, \tag{9.6}$$

but the solution $t_- = i\pi$ is not obtained now. In fact, the saddle point t_- vanishes as $z \downarrow 1$.

In Figure 11 we show the saddle point contours for $\beta = \frac{3}{4}$. For this value of β the saddle points coalesce when z equals $z_c = \frac{5}{3}$. We take $z = \frac{4}{3}$.

9.1.2 The oscillatory case: $z_c \le z$

We use the relations for the saddle point in (9.4) replacing p with $-ip = q$, where q is given in (6.15), and obtain

$$e^{t_\pm} = z\frac{(q\beta \pm 1)(i-\beta)}{q(\beta^2+1)} = z\frac{q\beta \pm 1}{q\sqrt{\beta^2+1}} e^{i(\pi - \arctan(1/\beta))}, \tag{9.7}$$

where we can replace $\pi - \arctan(1/\beta)$ with $\arccos(-\beta/\sqrt{1+\beta^2})$, and we obtain for the two saddle points

$$t_\pm = \ln \frac{z(q\beta \pm 1)}{q\sqrt{\beta^2+1}} + i \arccos \frac{-\beta}{\sqrt{1+\beta^2}}, \tag{9.8}$$

We see that the saddle points have the same imaginary parts. We have

$$t_+ + t_- = 2i \arccos \frac{-\beta}{\sqrt{1+\beta^2}}. \tag{9.9}$$

When $z \to \infty$, we have

$$t_\pm \sim \pm \ln \frac{2z\beta}{\sqrt{1+\beta^2}} + i \arccos \frac{-\beta}{\sqrt{1+\beta^2}}, \tag{9.10}$$

9.2 Integral representation for $K_{ia}(x)$

The modified Bessel function $K_{i\tau}(\mu\zeta)$ that is used in [2, p. 325] (with a slightly different notation) has the integral representation

$$K_{i\tau}(\mu\zeta) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-\mu\psi(w)}\, dw, \quad \psi(w) = \zeta \cosh w - i\beta w, \tag{9.11}$$

where $\beta = \tau/\mu$. We assume that $\zeta > 0$ and $\tau \geq 0$; μ is again a large positive parameter.The equation for the saddle points reads $\zeta \sinh w = i\beta$. When $\beta \leq \zeta$ we have the saddle points

$$w_+ = i \arcsin \frac{\beta}{\zeta}, \quad w_- = i\pi - i \arcsin \frac{\beta}{\zeta}. \tag{9.12}$$

When $\beta \geq \zeta$, we have

$$w_+ = \tfrac{1}{2}\pi i + \operatorname{arccosh}\frac{\beta}{\zeta}, \quad w_- = \tfrac{1}{2}\pi i - \operatorname{arccosh}\frac{\beta}{\zeta}. \tag{9.13}$$

There are more saddle points, but the given $w_\pm$ are relevant in our analysis.

We see that for $\beta/\zeta \leq 1$ the saddle points $w_\pm$ lie on the imaginary axis and when $\beta/\zeta \geq 1$ the become complex, and lie on the horizontal line with $\Im w_\pm = \frac{1}{2}\pi$.

Comparing the location of the saddle points $w_\pm$ for all ratios β/ζ with that of the saddle points $t_\pm$ for all ratios $z\beta/\sqrt{1+\beta^2}$ we observe that the pattern for the saddle points is quite similar in both cases. In both cases saddle points coalesce, and to obtain an asymptotic expansion that is valid for large μ when $t_+ \sim t_-$ or $w_+ \sim w_-$, Airy functions can be used . See [16] for more information on this topic from uniform asymptotic analysis for integrals, and for examples. See [4, 5] for using Airy-type expansions in numerical algorithms for the modified Bessel functions of pure imaginary order, and [6] for an extensive treatment of this type of expansions for computing several types of special functions.

In the present case we transform the integral (9.1) into an integral that is similar to that in (9.11), but with an extra function in the integrand. In this way we cover the case of coalescing saddle points, but also the large scale pattern of the saddle point behavior.

9.3 The transformation and the expansion

To obtain a representation of the conical $P-$function in terms of the modified Bessel function $K_{i\tau}(\mu\zeta)$ we use the following transformation of the $t-$variable in (9.1) to the $w-$variable in (9.11) by writing

$$\phi(t) = \psi(w) + \lambda, \tag{9.14}$$

where λ does not depend on t or w and should be determined, together with ζ in $\psi(w)$. This gives

$$P^{-\mu}_{-\frac{1}{2}+i\tau}(z) = \frac{\Gamma(\frac{1}{2}+\mu)\,(z^2-1)^{\mu/2}\,e^{-\mu\lambda}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_{-\infty}^{\infty} e^{-\mu\psi(w)} f(w)\,dw, \tag{9.15}$$

where

$$f(w) = \frac{dt}{dw}\,\frac{1}{\sqrt{z+\cosh t}}. \tag{9.16}$$

When we replace $f(w)$ in (9.15) with a constant the integral becomes the modified Bessel function given in (9.11). An asymptotic expansion can be obtained by using integration by parts. We put in the first step

$$f_0(w) = A_0(\beta,\zeta) + B_0(\beta,\zeta)\cosh w + \psi'(w) g_0(w), \quad f_0(w) = f(w), \tag{9.17}$$

where $A_0(\beta,\zeta)$ and $B_0(\beta,\zeta)$ follow from substituting $w = w_+$ and $w = w_-$. That is,

$$\begin{aligned} A_0(\beta,\zeta) &= \frac{f_0(w_-)\cosh w_+ - f_0(w_+)\cosh w_-}{\cosh w_+ - \cosh w_-}, \\ B_0(\beta,\zeta) &= \frac{f_0(w_+) - f_0(w_-)}{\cosh w_+ - \cosh w_-}. \end{aligned} \tag{9.18}$$

We denote the integral in (9.15) by J and replace $f(w)$ by the right-hand side of (9.17). This gives, by (9.11) and using integration by parts,

$$J = 2A_0(\beta,\zeta)K_{i\tau}(\mu\zeta) - 2B_0(\beta,\zeta)K'_{i\tau}(\mu\zeta) + \frac{1}{\mu}\int_{-\infty}^{\infty} e^{-\mu\psi(w)} f_1(w)\,dw, \tag{9.19}$$

where $f_1(w) = g_0'(w)$. This procedure can be continued, and we can obtain the representation (6.1), where the coefficients $A_n(\beta,\zeta)$ and $B_n(\beta,\zeta)$ of (6.4) follow from

$$f_n(w) = A_n(\beta,\zeta) + B_n(\beta,\zeta)\cosh w + \psi'(w) g_n(w), \quad n = 0,1,2,\ldots, \tag{9.20}$$

that is, from

$$
\begin{aligned}
A_n(\beta,\zeta) &= \frac{f_n(w_-)\cosh w_+ - f_n(w_+)\cosh w_-}{\cosh w_+ - \cosh w_-},\\
B_n(\beta,\zeta) &= \frac{f_n(w_+) - f_n(w_-)}{\cosh w_+ - \cosh w_-}.
\end{aligned}
\tag{9.21}
$$

The functions $f_n(w)$ follow from $f_{n+1}(w) = g_n'(w)$, $n \geq 0$.

9.4 Determination of λ and ζ

To determine λ and ζ we prescribe for the mapping in (9.14) that the saddle points in the $t-$plane should correspond with those in the $w-$plane. That is,

$$\phi(t_+) = \psi(w_+) + \lambda, \quad \phi(t_-) = \psi(w_-) + \lambda. \tag{9.22}$$

This gives

$$\zeta(\cosh w_+ - \cosh w_-) - i\beta(w_+ - w_-) = \phi(t_+) - \phi(t_-), \tag{9.23}$$

where $w_\pm$ also depend on ζ, see (9.12) and (9.13). When ζ is determined one of the equations in (9.22) can be used to determine λ. Because we have two different representations of the saddle points in (9.12) and (9.13) (and similar in (9.5) and (9.8), we obtain two different representations for ζ.

9.4.1 The monotonic case: $1 \leq z \leq z_c$, $\beta \leq \zeta$

In this case we have

$$\zeta\cosh w_+ = \sqrt{\zeta^2 - \beta^2}, \quad \zeta\cosh w_- = -\sqrt{\zeta^2 - \beta^2}, \tag{9.24}$$

and the left-hand side of (9.23) can be written as

$$2\left[\sqrt{\zeta^2 - \beta^2} - \beta\arccos(\beta/\zeta)\right]. \tag{9.25}$$

For the right-hand side we use (9.2) and (9.5). We have

$$z + \cosh t_+ = \frac{z(p+1)}{p(1+\beta^2)}, \quad z + \cosh t_- = \frac{z(p-1)}{p(1+\beta^2)}, \tag{9.26}$$

where p is given in (6.7). Hence

$$\phi(t_+) - \phi(t_-) = \ln\frac{p+1}{p-1} - \beta\arccos\frac{\beta^2p^2 - 1}{\beta^2p^2 + 1}. \tag{9.27}$$

Combining the two sides, we have the equation (6.6).

Next we determine λ from (9.22). Either equation can be used, but it is better to add the equations, and exploiting the symmetry. We easily find the value given in (6.2).

9.4.2 The oscillatory case: $z_c \le z,\ \zeta \le \beta$

Using (9.13) we obtain for the left-hand side of (9.23)

$$2i\left[\sqrt{\beta^2-\zeta^2}-\beta\operatorname{arccosh}(\beta/\zeta)\right]. \tag{9.28}$$

For the right-hand side we solve (9.3) . We have

$$\begin{aligned} z+\cosh t_+ &= \frac{z+i\sqrt{\beta^2(z^2-1)-1}}{\beta^2+1} = \sqrt{\frac{z^2-1}{\beta^2+1}}\,e^{i\operatorname{arccot} q},\\ z+\cosh t_- &= \frac{z-i\sqrt{\beta^2(z^2-1)-1}}{\beta^2+1} = \sqrt{\frac{z^2-1}{\beta^2+1}}\,e^{-i\operatorname{arccot} q}, \end{aligned} \tag{9.29}$$

where q is given in (6.15). This gives for the right-hand side we obtain, using (9.8)

$$\phi(t_+)-\phi(t_-) = 2i\operatorname{arccot} q - i\beta\ln\frac{\beta q+1}{\beta q-1}. \tag{9.30}$$

Combining the two sides, we obtain (6.14).

9.5 Representation of the coefficients for $\zeta \approx \beta$

The function $\Phi(\zeta)$ given in (6.3) is analytic for all $\zeta \ge 0$, in particular at $\zeta=\beta$ (and for complex values). We give expansions that can be used for representing $\Phi(\zeta)$ and the coefficients of the asymptotic expansions in (6.4).

From (6.6) we can obtain several expansions. First we mention

$$z = z_c + \sum_{k=1}^{\infty} z_k(\zeta-\beta)^k, \tag{9.31}$$

where the first two coefficients are

$$z_1 = \frac{-1}{\beta^2(1+\beta^2)^{1/6}},\quad z_2 = \frac{7+8\beta^2+3(1+\beta^2)^{2/3}}{10\beta^3(1+\beta^2)^{5/6}}. \tag{9.32}$$

We can use the expansion in (6.3) and obtain

$$\Phi^4(\zeta) = \frac{\beta^2}{(1+\beta^2)^{1/3}} + \frac{2\beta(3+2\beta^2+2(1+\beta^2)^{2/3})}{5(1+\beta^2)}(\zeta-\beta) + \mathcal{O}\left((\zeta-\beta)^2\right). \tag{9.33}$$

For representing the coefficients $A_n(\beta,\zeta), B_n(\beta,\zeta)$ of (6.4) it is convenient to expand $1/p$ in terms of powers of W. See (6.9). We write

$$\frac{1}{p} = \sum_{k=1}^{\infty} p_k W^k \tag{9.34}$$

and obtain, again from (6.6), $p_2 = p_4 = \ldots = 0$. Let

$$\gamma = \frac{1}{(1+\beta^2)^{1/3}}. \tag{9.35}$$

Then the first few odd coefficients are

$$\begin{aligned} p_1 = \gamma, \quad p_3 &= -\frac{\gamma^3(2\gamma^2+2\gamma+1)}{5(\gamma^2+\gamma+1)}, \\ p_5 &= -\frac{\gamma^5(37\gamma^4+74\gamma^3+69\gamma^2+27\gamma+3)}{175(\gamma^2+\gamma+1)^2}. \end{aligned} \tag{9.36}$$

For the coefficient $B_1(\beta,\zeta)$ given in (6.9) we have

$$\begin{aligned} B_1(\beta,\zeta) = &-\frac{\gamma\zeta(9\gamma^5+9\gamma^4+9\gamma^3+4\gamma+4)}{280(\gamma^2+\gamma+1)} + \\ &\frac{\gamma^3\zeta(98\gamma^7+196\gamma^6+213\gamma^5+83\gamma^4-47\gamma^3-96\gamma^2-76\gamma-56)}{12600(\gamma^2+\gamma+1)^2}W^2 + \\ &\mathcal{O}(W^4). \end{aligned} \tag{9.37}$$

Observe that we have not expanded the quantity ζ that appears in all terms as a linear factor. This expansion holds for $1 \le z \le z_c$, that is, $\zeta \ge \beta$. When $z \ge z_c$ we have the same expansion with W^2 replaced with $-W^2$, throughout.

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